Solución Examen Final A

1 a) $f(t) = \begin{cases} 2 & \text{si } 0 \leq t < 3 \\ -2 & \text{si } t \geq 3 \end{cases}$

se puede expresar como $f(t) = 2 + (-2-2)u(t-3)$, o

$f(t) = 2 - 4u(t-3)$. Tomando transf. de Laplace:

$$\boxed{\mathcal{L}\{f(t)\} = \frac{2}{s} - \frac{4}{s} e^{-3s}}$$

b) $$\frac{4}{s(s^2+2s+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+4}$$

$$\Rightarrow \quad 4 = A(s^2+2s+4) + Bs^2 + Cs$$

$$\Rightarrow \quad 4 = (A+B)s^2 + (2A+C)s + 4A$$

$$\Rightarrow \quad \begin{aligned} A+B &= 0 \\ 2A+C &= 0 \\ 4A &= 4 \end{aligned} \Rightarrow \boxed{B=-1} \quad \boxed{C=-2} \quad \boxed{A=1}$$

Por tanto $$\frac{4}{s(s^2+2s+4)} = \frac{1}{s} - \frac{s+2}{s^2+2s+4}$$

$$= \frac{1}{s} - \frac{s+2}{(s^2+2s+1)+3} = \frac{1}{s} - \frac{s+2}{(s+1)^2+3}$$

$$= \frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+3}$$

$$= \frac{1}{s} - \frac{s+1}{(s+1)^2+(\sqrt{3})^2} - \frac{1}{\sqrt{3}} \frac{\sqrt{3}}{(s+1)^2+(\sqrt{3})^2}$$

De la expresión anterior obtenemos que

$$\mathcal{L}^{-1}\left\{\frac{4}{s(s^2+2s+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{3})^2}\right\} - \frac{1}{\sqrt{3}}e^{-t}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{s^2+(\sqrt{3})^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{4}{s(s^2+2s+4)}\right\} = 1 - e^{-t}\cos(\sqrt{3}t) - \frac{1}{\sqrt{3}}e^{-t}\operatorname{sen}(\sqrt{3}t).$$

c) $\mathcal{L}^{-1}\left\{ \frac{4}{s(s^2+2s+4)} e^{-3s} \right\}$

$$= u(t-3)\, \mathcal{L}^{-1}\left\{ \frac{4}{s(s^2+2s+4)} \right\}\Big|_{t \mapsto t-3}$$

$$= u(t-3)\left[1 - e^{-t}\cos(\sqrt{3}t) - \frac{1}{\sqrt{3}} e^{-t}\operatorname{sen}(\sqrt{3}t)\right]\Big|_{t \mapsto t-3}$$

$$= u(t-3)\left[1 - e^{-(t-3)}\cos\left(\sqrt{3}(t-3)\right) - \frac{1}{\sqrt{3}} e^{-(t-3)}\operatorname{sen}\left(\sqrt{3}(t-3)\right)\right].$$

d) Sea $x(t)$ la posición de la masa en ~~del~~ el tiempo t. Los datos dados son los siguientes:

$m = 1\ kg$ $\qquad k = ?$

$x(0) = 0$ $\qquad W = ?$

$x'(0) = 0$ $\qquad W = mg = 10\ N \Rightarrow k = \frac{W}{s} = \frac{10/1}{5/2}$

$\beta = 2$ $\qquad \Rightarrow \boxed{k = 4}$

$s = 5/2$

Planteamiento del problema de valor inicial:

$$\begin{cases} x'' + 2x' + 4x = 2 - 4u(t-3) = f(t) \\ x(0) = 0 \\ x'(0) = 0. \end{cases}$$

solución del problema de valor inicial:

$$\mathcal{L}\{x''\} + 2\mathcal{L}\{x'\} + 4\mathcal{L}\{x\} = \mathcal{L}\{f(t)\}$$

$$s^2X(s) - s\cancel{x(0)}^{0} - \cancel{x'(0)}^{0} + 2\left[sX(s) - \cancel{x(0)}^{0}\right] + 4X(s) = \frac{2}{s} - \frac{4}{s}e^{-3s}$$

$$(s^2+2s+4)\,X(s) = \frac{2}{s} - \frac{4}{s}e^{-3s}$$

$$X(s) = \frac{2}{s(s^2+2s+4)} - \frac{4}{s(s^2+2s+4)}e^{-3s}$$

$$X(s) = \frac{1}{2}\,\frac{4}{s(s^2+2s+4)} - \frac{4}{s(s^2+2s+4)}e^{-3s}$$

De los incisos b) y c) obtenemos que

$$x(t) = \frac{1}{2}\left[1 - e^{-t}\cos(\sqrt{3}t) - e^{-t}\operatorname{sen}(\sqrt{3}t)\right]$$
$$- \left[1 - e^{-(t-3)}\cos(\sqrt{3}(t-3)) - e^{-(t-3)}\operatorname{sen}(\sqrt{3}(t-3))\right]\mathcal{U}(t-3).$$

2. $f'(t) = 1 - 4\int_0^t f(\tau)d\tau \;,\; f(0) = 4$

$$\mathcal{L}\{f'(t)\} = \mathcal{L}\{1\} - 4\,\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}$$

$$sF(s) - f(0) = \frac{1}{s} - 4.\,\frac{F(s)}{s}$$

$$sF(s) + \frac{4}{s}F(s) = f(0) + \frac{1}{s} = 4 + \frac{1}{s}$$

$$\left(s + \frac{4}{s}\right)F(s) = 4 + \frac{1}{s}$$

$$\frac{s^2+4}{s}\cdot F(s) = 4 + \frac{1}{s} \implies F(s) = \frac{4s}{s^2+4} + \frac{1}{s^2+4} = 4.\frac{s}{s^2+4} + \frac{1}{2}\,\frac{2}{s^2+4}$$

$$\implies f(t) = 4\cos(2t) + \frac{1}{2}\operatorname{sen}(2t).$$