

Nombre:
Grupo :

Tema: Integración por partes
 Sabiendo que

$$F(x) = u(x)v(x) \rightarrow F'(x) = v(x)u'(x) + u(x)v'(x)$$

Tenemos:

$$F'(x) = v(x)u'(x) + u(x)v'(x) \rightarrow$$

$$\frac{dF}{dx} = v(x)\frac{du}{dx} + u(x)\frac{dv}{dx} \rightarrow$$

$$dF = v(x) du + u(x) dv \rightarrow$$

$$\int dF = \int (v(x) du + u(x) dv) \rightarrow$$

$$F(x) = \int v(x) du + \int u(x) dv \rightarrow$$

$$\int u(x) dv = F(x) - \int v(x) du \rightarrow$$

$$\int u(x) dv = u(x)v(x) - \int v(x) du \rightarrow$$

$$\int u dv = uv - \int v du$$

Por lo tanto

$$\int u dv = uv - \int v du$$

Fórmula para la integración por partes

Aplique la integración por partes para verificar las siguientes integrales

1. Forma $\int x^k \ln(x) dx$

$$\text{Sea } \begin{cases} u = \ln x \\ du = \frac{dx}{x} \end{cases} \quad \begin{matrix} + \\ \searrow \\ \leftarrow \\ - \end{matrix} \quad \begin{cases} dv = x^k dx \\ v = \frac{x^{k+1}}{k+1} \end{cases}, \text{ entonces}$$

$$\begin{aligned} \int x^k \ln(x) dx &= \frac{x^{k+1}}{k+1} \ln x - \int \frac{x^{k+1}}{k+1} \frac{dx}{x} \\ &= \frac{x^{k+1}}{k+1} \ln x - \frac{1}{k+1} \int x^k dx \end{aligned}$$

Verifique, utilizando la reducción a la forma 1, el resultado en las siguientes integrales

(a)

$$\int \sqrt{x} \ln(x) dx = \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{4}{9} x^{\frac{3}{2}} + C$$

(b)

$$\int \frac{\ln(x)}{x^3} dx = -\frac{\ln x}{2x^2} + \frac{1}{4x^2} + C$$

(c)

$$\int \frac{\ln(x)}{\sqrt[3]{x}} dx = \frac{3}{4} x^{\frac{4}{3}} \ln x - \frac{9}{16} x^{\frac{4}{3}} + C$$

(d)

$$\int 3x^2 \ln(2x) dx = \frac{1}{3} x^3 (3 \ln 2 + 3 \ln x - 1) + K$$

2. Forma $\int x^k \ln^m(x) dx$

$$\text{Sea } \left\{ \begin{array}{ccc} u = \ln^m x & & dv = x^k dx \\ & + & \\ & \searrow & \\ du = m \ln^{m-1} x \frac{dx}{x} & \leftarrow & v = \frac{x^{k+1}}{k+1} \end{array} \right. , \text{ entonces}$$

$$\begin{aligned} \int x^k \ln^m(x) dx &= \frac{x^{k+1}}{k+1} \ln^m x - \int \frac{mx^{k+1}}{k+1} \ln^{m-1} x \frac{dx}{x} \\ &= \frac{x^{k+1}}{k+1} \ln^m x - \frac{m}{k+1} \int x^k \ln^{m-1} x dx \end{aligned}$$

Ejemplo: $\int \ln^2 x dx$

$$\text{Sea } \left\{ \begin{array}{ccc} u = \ln^2 x & & dv = dx \\ & + & \\ & \searrow & \\ du = 2 \ln x \frac{dx}{x} & \leftarrow & v = x \end{array} \right. , \text{ entonces}$$

$$\int \ln^2 x dx = x \ln^2 x - 2 \int \ln x dx$$

$$\begin{aligned} \text{Sea } \left\{ \begin{array}{ccc} u = \ln x & & dv = dx \\ & + & \\ & \searrow & \\ du = \frac{dx}{x} & \leftarrow & v = x \end{array} \right. , \text{ entonces} \\ \int \ln^2 x dx &= x \ln^2 x - 2(x \ln x - \int dx) \\ &= x \ln^2 x - 2(x \ln x - x) + K \\ &= x \ln^2 x - 2x \ln x + 2x + K \end{aligned}$$

Determine $\int x^3 \ln^2 x dx$

$$\text{Sea } \left\{ \begin{array}{ccc} u = \ln^2 x & & dv = x^3 dx \\ & (+) & \\ & \searrow & \\ du = \frac{2 \ln x}{x} & \leftarrow & v = \frac{x^4}{4} \end{array} \right. , \text{ entonces}$$

$$\int x^3 \ln^2 x dx = \frac{x^4}{4} \ln^2 x - \frac{1}{2} \int x^3 \ln x dx$$

$$\text{Sea } \left\{ \begin{array}{ccc} u = \ln x & & dv = x^3 dx \\ & (+) & \\ & \searrow & \\ du = \frac{1}{x} & \leftarrow & v = \frac{x^4}{4} \end{array} \right.$$

$$\begin{aligned}
\int x^3 \ln^2 x dx &= \frac{x^4}{4} \ln^2 x - \frac{1}{2} \left(\frac{x^4}{4} \ln x - \int x^3 dx \right) \\
&= \frac{x^4}{4} \ln^2 x - \frac{1}{2} \left(\frac{x^4}{4} \ln x - \frac{x^4}{4} \right) \\
&= \frac{x^4}{4} \ln^2 x - \frac{x^4}{8} \ln x + \frac{x^4}{8} + K
\end{aligned}$$

Utilizando la reducción a la forma 2, pruebe que

(a)

$$\int \ln^3 x dx = x \ln^3 x - 3x \ln^2 x + 6x \ln x - 6x + K$$

(b)

$$\int x^2 \ln^2 x dx = \frac{1}{27} x^3 (9 \ln^2 x - 6 \ln x + 2) + K$$

(c)

$$\int (2x+1) \ln^2(x) dx = x^2 \ln^2 x - x^2 \ln x + \frac{1}{2} x^2 + x \ln^2 x - 2x \ln x + 2x + K$$

3. Forma $\int x^k e^{mx} dx$

$$\text{Sea } \left\{ \begin{array}{ll} u = x^k & dv = e^{mx} dx \\ & \begin{array}{c} (+) \\ \searrow \\ \leftarrow \\ (-) \end{array} \\ du = kx^{k-1} dx & v = \frac{1}{m} e^{mx} \end{array} \right. , \text{ entonces}$$

$$\begin{aligned}
\int x^k e^{mx} dx &= \frac{x^k}{m} e^{mx} - \int \frac{kx^{k-1}}{m} e^{mx} dx \\
&= \frac{x^k}{m} e^{mx} - \frac{k}{m} \int x^{k-1} e^{mx} dx
\end{aligned}$$

El proceso se repite en caso de ser necesario, con la integral que queda a la derecha.

Ejemplo: Determine $\int 5x e^{3x} dx$

$$\text{Sea } \left\{ \begin{array}{ll} u = 5x & dv = e^{3x} dx \\ & \begin{array}{c} (+) \\ \searrow \\ \leftarrow \\ (-) \end{array} \\ du = 5 dx & v = \frac{1}{3} e^{3x} \end{array} \right. \text{ entonces}$$

$$\begin{aligned}
\int 5x e^{3x} dx &= \frac{5}{3} x e^{3x} - \frac{10}{3} \int e^{3x} dx \\
&= \frac{5}{3} x e^{3x} - \frac{5}{3} \left(\frac{1}{3} e^{3x} \right) + C \\
&= \frac{5}{3} x e^{3x} - \frac{5}{9} e^{3x} + C
\end{aligned}$$

Determine

$$\int 5x^2 e^{3x} dx$$

$$Sea \left\{ \begin{array}{ll} u = 5x^2 & dv = e^{3x} dx \\ & (+) \searrow \\ du = 10x dx & \leftarrow (-) \quad v = \frac{1}{3} e^{3x} \end{array} \right. , \text{ entonces}$$

$$\int 5x^2 e^{3x} dx = \frac{5}{3} x^2 e^{3x} - \frac{1}{3} \int 10x e^{3x} dx$$

$$Sea \left\{ \begin{array}{ll} u = 10x & dv = e^{3x} dx \\ & (+) \searrow \\ du = 10 dx & \leftarrow (-) \quad v = \frac{1}{3} e^{3x} \end{array} \right. , \text{ entonces}$$

$$\int 5x^2 e^{3x} dx = \frac{5}{3} x^2 e^{3x} - \frac{1}{3} \left(\frac{10}{3} x e^{3x} - \frac{10}{3} \int e^{3x} dx \right)$$

$$= \frac{5}{3} x^2 e^{3x} - \frac{10}{9} x e^{3x} + \frac{10}{9} \left(\frac{1}{3} e^{3x} \right) + C$$

$$= \frac{5}{3} x^2 e^{3x} - \frac{10}{9} x e^{3x} + \frac{10}{27} e^{3x} + C$$

Ejemplo: Pruebe que, utilizando la integración tabular

$$\int 5x^2 e^{3x} dx = \frac{5}{3} x^2 e^{3x} - \frac{10}{9} x e^{3x} + \frac{10}{27} e^{3x} + C$$

Derivar $p(x) = 5x^2$		Integrar $f(x) = e^{3x}$
	$\searrow$ +	
10x		$\frac{1}{3} e^{3x}$
	$\searrow$ -	
10		$\frac{1}{9} e^{3x}$
	$\searrow$ +	
0		$\frac{1}{27} e^{3x}$

luego

$$\int 5x^2 e^{3x} dx = \frac{5}{3} x^2 e^{3x} - \frac{10}{9} x e^{3x} + \frac{10}{27} e^{3x} + C$$

Verifique, utilizando la integración tabular que

(a)

$$\int 7x^2 e^{-4x} dx = \frac{7}{8} \frac{x}{e^{4x}} + \frac{7}{4} \frac{x^2}{e^{4x}} + \frac{7}{32e^{4x}} + K$$

(b)

$$\int (x^2 + 2x + 1)e^{2x} dx = \frac{1}{2}x^2 e^{2x} + \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} + K$$

(c)

$$\int (x^2 + x + 1)e^{3x+1} dx = \frac{1}{27}e^{3x+1} (9x^2 + 3x + 8) + K$$

4. Forma $\int x^k(mx+b)^{\frac{p}{q}} dx$

$$\text{Sea } \begin{cases} u = x^k & dv = (mx+b)^{\frac{p}{q}} dx \\ \quad \quad \quad \begin{matrix} (+) \\ \searrow \\ \leftarrow \\ (-) \end{matrix} & v = \frac{q}{(p+q)m} (mx+b)^{\frac{p}{q}+1} \\ du = kx^{k-1} dx & \end{cases}$$

Entonces

$$\begin{aligned} \int x^k (mx+b)^{\frac{p}{q}} dx &= \frac{qx^k}{(p+q)m} (mx+b)^{\frac{p}{q}+1} - \int kx^{k-1} \frac{q}{(p+q)m} (mx+b)^{\frac{p}{q}+1} dx \\ &= \frac{qx^k}{(p+q)m} (mx+b)^{\frac{p}{q}+1} - \frac{kq}{(p+q)m} \int x^{k-1} (mx+b)^{\frac{p}{q}+1} dx \end{aligned}$$

Ejemplo.

Determine $\int 2x(3x+2)^{\frac{1}{2}} dx$

$$\text{Sea } \begin{cases} u = 2x & dv = (3x+2)^{\frac{1}{2}} dx \\ \quad \quad \quad \begin{matrix} (+) \\ \searrow \\ \leftarrow \\ (-) \end{matrix} & v = \frac{2}{9} (3x+2)^{\frac{3}{2}} \\ du = 2dx & \end{cases}, \text{ entonces}$$

$$\begin{aligned} \int 2x(3x+2)^{\frac{1}{2}} dx &= \frac{4}{9}x(3x+2)^{\frac{3}{2}} - \frac{4}{9} \int (3x+2)^{\frac{3}{2}} dx \\ &= \frac{4}{9}x(3x+2)^{\frac{3}{2}} - \frac{4}{9} \frac{2}{5} \frac{1}{3} (3x+2)^{\frac{5}{2}} dx \\ &= \frac{4}{9}x(3x+2)^{\frac{3}{2}} - \frac{8}{135} (3x+2)^{\frac{5}{2}} + K \end{aligned}$$

Puede resolverse tambien utilizando el método de sustitución

$$\int 2x(3x+2)^{\frac{1}{2}} dx$$

$$\text{Sea } \begin{cases} u = 3x+2 \rightarrow x = \frac{u}{3} - \frac{2}{3} \\ du = 3dx \end{cases}, \text{ entonces}$$

$$\begin{aligned}
\int 2x(3x+2)^{\frac{1}{2}} dx &= \frac{2}{3} \int x(3x+2)^{\frac{1}{2}} (3dx) \\
&= \frac{2}{3} \int \left(\frac{u}{3} - \frac{2}{3}\right) u^{\frac{1}{2}} du \\
&= \frac{2}{3} \int \left(\frac{u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{1}{2}}}{3}\right) du \\
&= \frac{2}{3} \frac{1}{3} \frac{2}{5} u^{\frac{5}{2}} - \frac{4}{3} \frac{1}{3} \frac{2}{3} u^{\frac{3}{2}} \\
&= \frac{4}{45} (3x+2)^{\frac{5}{2}} - \frac{8}{27} (3x+2)^{\frac{3}{2}} + K
\end{aligned}$$

Otro ejemplo, por sustitución

$$\begin{aligned}
&\int 5x^2(2x+1)^{\frac{1}{3}} dx \\
\text{Sea } \begin{cases} u = 2x+1 & \rightarrow & x = \frac{u}{2} - \frac{1}{2} \\ du = 2dx \end{cases} \quad \text{entonces}
\end{aligned}$$

$$\begin{aligned}
\int 5x^2(2x+1)^{\frac{1}{3}} dx &= \frac{5}{2} \int x^2(2x+1)^{\frac{1}{3}} (2dx) \\
&= \frac{5}{2} \int \left(\frac{u}{2} - \frac{1}{2}\right)^2 u^{\frac{1}{3}} du \\
&= \frac{2}{3} \frac{1}{4} \int (u - 1)^2 u^{\frac{1}{3}} du \\
&= \frac{1}{6} \int (u^2 - 2u + 1) u^{\frac{1}{3}} du \\
&= \frac{1}{6} \int (u^{\frac{7}{3}} - 2u^{\frac{4}{3}} + u^{\frac{1}{3}}) du \\
&= \frac{1}{6} \left(\frac{3}{10} u^{\frac{10}{3}} - 2 \frac{3}{7} u^{\frac{7}{3}} + \frac{3}{4} u^{\frac{4}{3}} \right) + K \\
&= \frac{1}{20} (2x+1)^{\frac{10}{3}} - \frac{1}{7} (2x+1)^{\frac{7}{3}} + \frac{1}{8} (2x+1)^{\frac{4}{3}} + K
\end{aligned}$$

Resolver utilizando el método de sustitución

- (a) $\int x(2x-3)^{\frac{1}{3}} dx$
- (b) $\int x(2x-3)^{-\frac{1}{3}} dx$
- (c) $\int x^2(3x-1)^{\frac{1}{2}} dx$
- (d) $\int x^2(3x-1)^{-\frac{1}{2}} dx$
- (e) $\int (2x+1)^2(3x-1)^{\frac{1}{2}} dx$
- (f) $\int (2x+1)^2(3x-1)^{-\frac{1}{2}} dx$

Utilizando integración por partes (integración tabular)

$$\int 5x^2(2x+1)^{\frac{1}{3}} dx$$

Derivar		Integrar
$p(x) = 5x^2$		$f(x) = (2x+1)^{\frac{1}{3}}$
	$\searrow$	
	+	
$10x$		$\frac{1}{2} \frac{3}{4} (2x+1)^{\frac{4}{3}}$
	$\searrow$	
	-	
10		$\frac{3}{8} \frac{1}{2} \frac{3}{7} (2x+1)^{\frac{7}{3}}$
	$\searrow$	
	+	
0		$\frac{9}{112} \frac{1}{2} \frac{3}{10} (2x+1)^{\frac{10}{3}}$

entonces

$$\int 5x^2(2x+1)^{\frac{1}{3}} dx = \frac{15}{8} x^2(2x+1)^{\frac{4}{3}} - \frac{90}{112} x(2x+1)^{\frac{7}{3}} + \frac{27}{224} (2x+1)^{\frac{10}{3}} + K$$

Resolver utilizando el método de integración por partes (Integración tabular)

- (a) $\int x(2x-3)^{\frac{1}{3}} dx$
- (b) $\int x(2x-3)^{-\frac{1}{3}} dx$
- (c) $\int x^2(3x-1)^{\frac{1}{2}} dx$
- (d) $\int x^2(3x-1)^{-\frac{1}{2}} dx$
- (e) $\int (2x+1)^2(3x-1)^{\frac{1}{2}} dx$
- (f) $\int (2x+1)^2(3x-1)^{-\frac{1}{2}} dx$