UNIVERSIDAD DEL NORTE

Departamento de Matemáticas y Estadísticas
Actividad de formación Integral
Cálculo II ANEC

Nombre:	**Grupo :**

Tema: Integración por partes

Sabiendo que

$$F(x) = u(x)v(x) \rightarrow F'(x) = v(x)u'(x) + u(x)v'(x)$$

Tenemos:

$$\begin{aligned}
F'(x) &= v(x)u'(x) + u(x)v'(x) && \rightarrow \\
\frac{dF}{dx} &= v(x)\frac{du}{dx} + u(x)\frac{dv}{dx} && \rightarrow \\
dF &= v(x)\ du\ + u(x)\ dv && \rightarrow \\
\int dF &= \int (v(x)\ du\ + u(x)\ dv\ ) && \rightarrow \\
F(x) &= \int v(x)\ du\ + \int u(x)\ dv && \rightarrow \\
\int u(x)\ dv &= F(x) - \int v(x)\ du && \rightarrow \\
\int u(x)\ dv &= u(x)v(x) - \int v(x)\ du && \rightarrow \\
\int u\ dv &= uv - \int v\ du
\end{aligned}$$

Por lo tanto

$$\int u\ dv\ = uv - \int v\ du$$

Fórmula para la integración por partes

Aplique la integración por partes para verificar las siguientes integrales

1. Forma $\int x^k \ln(x)dx$

   $$\text{Sea} \begin{cases} u = \ln x & \qquad + \searrow \qquad & dv = x^k dx \\ du = \dfrac{dx}{x} & \qquad \leftarrow \ - \qquad & v = \dfrac{x^{k+1}}{k+1} \end{cases}, \text{ entonces}$$

   $$\begin{aligned}
   \int x^k \ln(x)dx &= \frac{x^{k+1}}{k+1}\ln x - \int \frac{x^{k+1}}{k+1}\frac{dx}{x} \\
   &= \frac{x^{k+1}}{k+1}\ln x - \frac{1}{k+1}\int \ x^k \quad dx
   \end{aligned}$$

   Verifique, utilizando la reducción a la forma 1, el resultado en las siguientes integrales

   (a)
   $$\int \sqrt{x}\ln(x)dx = \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + C$$

   (b)
   $$\int \frac{\ln(x)}{x^3}dx = -\frac{\ln x}{2x^2} + \frac{1}{4x^2} + C$$

(c)

$$\int \frac{\ln(x)}{\sqrt[3]{x}}dx = \frac{3}{4}x^{\frac{4}{3}}\ln x - \frac{9}{16}x^{\frac{4}{3}} + C$$

(d)

$$\int 3x^2\ln(2x)dx = \frac{1}{3}x^3\left(3\ln 2 + 3\ln x - 1\right) + K$$

2. Forma $\int x^k \ln^m(x)dx$

$$\text{Sea}\begin{cases} u = \ln^m x & & dv = x^k dx \\ & + & \\ & \searrow & \\ du = m\ln^{m-1} x\dfrac{dx}{x} & \leftarrow & v = \dfrac{x^{k+1}}{k+1} \\ & - & \end{cases}, \text{ entonces}$$

$$\begin{aligned} \int x^k \ln^m(x)dx &= \frac{x^{k+1}}{k+1}\ln^m x - \int \frac{mx^{k+1}}{k+1}\ln^{m-1} x\frac{dx}{x} \\ &= \frac{x^{k+1}}{k+1}\ln^m x - \frac{m}{k+1}\int x^k \ln^{m-1} xdx \end{aligned}$$

Ejemplo: $\int \ln^2 xdx$

$$Sea\begin{cases} u = \ln^2 x & & dv = dx \\ & + & \\ & \searrow & \\ du = 2\ln x\dfrac{dx}{x} & \leftarrow & v = x \\ & - & \end{cases}, \text{ entonces}$$

$$\int \ln^2 xdx = x\ln^2 x - 2\int \ln xdx$$

$$\text{Sea}\begin{cases} u = \ln x & & dv = dx \\ & + & \\ & \searrow & \\ du = \dfrac{dx}{x} & \leftarrow & v = x \\ & - & \end{cases}, \text{entonces}$$

$$\begin{aligned} \int \ln^2 xdx &= x\ln^2 x - 2(x\ln x - \int dx) \\ &= x\ln^2 x - 2(x\ln x - x) + K \\ &= x\ln^2 x - 2x\ln x + 2x + K \end{aligned}$$

Determine $\int x^3 \ln^2 xdx$

$$Sea\begin{cases} u = \ln^2 x & & dv = x^3dx \\ & (+) & \\ & \searrow & \\ du = \dfrac{2\ln x}{x} & \leftarrow & v = \dfrac{x^4}{4} \\ & (-) & \end{cases}, entonces$$

$$\int x^3 \ln^2 xdx = \frac{x^4}{4}\ln^2 x - \frac{1}{2}\int x^3 \ln xdx$$

$$Sea\begin{cases} u = \ln x & & dv = x^3dx \\ & (+) & \\ & \searrow & \\ du = \dfrac{1}{x} & \leftarrow & v = \dfrac{x^4}{4} \\ & (-) & \end{cases}$$

$$\begin{aligned}\int x^3 \ln^2 x dx &= \frac{x^4}{4}\ln^2 x - \frac{1}{2}(\frac{x^4}{4}\ln x - \int x^3 dx)\\ &= \frac{x^4}{4}\ln^2 x - \frac{1}{2}(\frac{x^4}{4}\ln x - \frac{x^4}{4})\\ &= \frac{x^4}{4}\ln^2 x - \frac{x^4}{8}\ln x + \frac{x^4}{8} + K\end{aligned}$$

Utilizando la reducción a la forma 2, pruebe que

(a)

$$\int \ln^3 x dx = x\ln^3 x - 3x\ln^2 x + 6x\ln x - 6x + K$$

(b)

$$\int x^2 \ln^2 x dx = \frac{1}{27}x^3\left(9\ln^2 x - 6\ln x + 2\right) + K$$

(c)

$$\int (2x+1)\ln^2(x)dx = x^2\ln^2 x - x^2\ln x + \frac{1}{2}x^2 + x\ln^2 x - 2x\ln x + 2x + K$$

3. Forma $\int x^k e^{mx} dx$

$$Sea \begin{cases} u = x^k & \overset{(+)}{\searrow} \quad dv = e^{mx}dx \\ du = kx^{k-1}dx & \underset{(-)}{\leftarrow} \quad v = \frac{1}{m}e^{mx} \end{cases}, entonces$$

$$\begin{aligned}\int x^k e^{mx} dx &= \frac{x^k}{m}e^{mx} - \int \frac{kx^{k-1}}{m}e^{mx}dx\\ &= \frac{x^k}{m}e^{mx} - \frac{k}{m}\int x^{k-1} e^{mx}dx\end{aligned}$$

El proceso se repite en caso de ser necesario, con la integral que queda a la derecha.

Ejemplo: Determine $\int 5xe^{3x}dx$

$$Sea \begin{cases} u = 5x & \overset{(+)}{\searrow} \quad dv = e^{3x}dx \\ du = 5dx & \underset{(-)}{\leftarrow} \quad v = \frac{1}{3}e^{3x} \end{cases} \text{ entonces}$$

$$\begin{aligned}\int 5xe^{3x}dx &= \frac{5}{3}xe^{3x} - \frac{10}{3}\int e^{3x}dx\\ &= \frac{5}{3}xe^{3x} - \frac{5}{3}(\frac{1}{3}e^{3x}) + C\\ &= \frac{5}{3}xe^{3x} - \frac{5}{9}e^{3x} + C\end{aligned}$$

Determine

$\int 5x^2e^{3x}dx$

$$Sea \left\{ \begin{array}{ccc} u = 5x^2 & & dv = e^{3x}dx \\ & (+) & \\ & \searrow & \\ du = 10xdx & \leftarrow & v = \frac{1}{3}e^{3x} \\ & (-) & \end{array} \right. \text{, entonces}$$

$$\int 5x^2e^{3x}dx = \frac{5}{3}x^2e^{3x} - \frac{1}{3}\int 10xe^{3x}dx$$

$$Sea \left\{ \begin{array}{ccc} u = 10x & & dv = e^{3x}dx \\ & (+) & \\ & \searrow & \\ du = 10dx & \leftarrow & v = \frac{1}{3}e^{3x} \\ & (-) & \end{array} \right. \text{,entonces}$$

$$\begin{aligned} \int 5x^2e^{3x}dx &= \frac{5}{3}x^2e^{3x} - \frac{1}{3}(\frac{10}{3}xe^{3x} - \frac{10}{3}\int e^{3x}dx) \\ &= \frac{5}{3}x^2e^{3x} - \frac{10}{9}xe^{3x} + \frac{10}{9}(\frac{1}{3}e^{3x}) + C \\ &= \frac{5}{3}x^2e^{3x} - \frac{10}{9}xe^{3x} + \frac{10}{27}e^{3x} + C \end{aligned}$$

Ejemplo: Pruebe que, utilizando la integración tabular

$$\int 5x^2e^{3x}dx = \frac{5}{3}x^2e^{3x} - \frac{10}{9}xe^{3x} + \frac{10}{27}e^{3x} + C$$

Derivar		Integrar
$p(x) = 5x^2$		$f(x) = e^{3x}$
	$\searrow$ $+$	
$10x$		$\frac{1}{3}e^{3x}$
	$\searrow$ $-$	
10		$\frac{1}{9}e^{3x}$
	$\searrow$ $+$	
0		$\frac{1}{27}e^{3x}$

luego

$$\int 5x^2e^{3x}dx = \frac{5}{3}x^2e^{3x} - \frac{10}{9}xe^{3x} + \frac{10}{27}e^{3x} + C$$

Verifique, utilizando la integración tabular que

(a)

$$\int 7x^2e^{-4x}dx = \frac{7}{8}\frac{x}{e^{4x}} + \frac{7}{4}\frac{x^2}{e^{4x}} + \frac{7}{32e^{4x}} + K$$

(b)

$$\int (x^2+2x+1)e^{2x}dx = \frac{1}{2}x^2e^{2x} + \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + K$$

(c)

$$\int (x^2+x+1)e^{3x+1}dx = \frac{1}{27}e^{3x+1}\left(9x^2+3x+8\right) + K$$

4. Forma $\int x^k(mx+b)^{\frac{p}{q}}dx$

$$\text{Sea}\begin{cases} u = x^k & \overset{(+)}{\searrow} \quad dv = (mx+b)^{\frac{p}{q}}dx \\ du = kx^{k-1}dx & \underset{(-)}{\leftarrow} \quad v = \dfrac{q}{(p+q)m}(mx+b)^{\frac{p}{q}+1} \end{cases}$$

Entonces

$$\begin{aligned} \int x^k(mx+b)^{\frac{p}{q}}dx &= \frac{qx^k}{(p+q)m}(mx+b)^{\frac{p}{q}+1} - \int kx^{k-1}\frac{q}{(p+q)m}(mx+b)^{\frac{p}{q}+1}dx \\ &= \frac{qx^k}{(p+q)m}(mx+b)^{\frac{p}{q}+1} - \frac{kq}{(p+q)m}\int x^{k-1}(mx+b)^{\frac{p}{q}+1}dx \end{aligned}$$

Ejemplo.
Determine $\int 2x(3x+2)^{\frac{1}{2}}dx$

$$Sea\begin{cases} u = 2x & \overset{(+)}{\searrow} \quad dv = (3x+2)^{\frac{1}{2}}dx \\ du = 2dx & \underset{(-)}{\leftarrow} \quad v = \dfrac{2}{9}(3x+2)^{\frac{3}{2}} \end{cases}, entonces$$

$$\begin{aligned} \int 2x(3x+2)^{\frac{1}{2}}dx &= \frac{4}{9}x(3x+2)^{\frac{3}{2}} - \frac{4}{9}\int (3x+2)^{\frac{3}{2}}dx \\ &= \frac{4}{9}x(3x+2)^{\frac{3}{2}} - \frac{4}{9}\frac{2}{5}\frac{1}{3}(3x+2)^{\frac{5}{2}}dx \\ &= \frac{4}{9}x(3x+2)^{\frac{3}{2}} - \frac{8}{135}(3x+2)^{\frac{5}{2}} + K \end{aligned}$$

Puede resolverse tambien utilizando el método de sustitución

$$\int 2x(3x+2)^{\frac{1}{2}}dx$$

$$Sea\begin{cases} u = 3x+2 \quad \rightarrow \quad x = \dfrac{u}{3} - \dfrac{2}{3} \\ du = 3dx \end{cases}, \text{ entonces}$$

$$
\begin{aligned}
\int 2x(3x+2)^{\frac{1}{2}}dx &= \tfrac{2}{3}\int x(3x+2)^{\frac{1}{2}}(3dx)\\
&= \frac{2}{3}\int(\frac{u}{3}-\frac{2}{3})u^{\frac{1}{2}}du\\
&= \frac{2}{3}\int(\frac{u^{\frac{3}{2}}}{3}-\frac{2u^{\frac{1}{2}}}{3})du\\
&= \frac{2}{3}\frac{1}{3}\frac{2}{5}u^{\frac{5}{2}}-\frac{4}{3}\frac{1}{3}\frac{2}{3}u^{\frac{3}{2}}\\
&= \frac{4}{45}(3x+2)^{\frac{5}{2}}-\frac{8}{27}(3x+2)^{\frac{3}{2}}+K
\end{aligned}
$$

Otro ejemplo, por sustitución

$$\int 5x^2(2x+1)^{\frac{1}{3}}dx$$

$$
Sea\begin{cases} u=2x+1 \quad \rightarrow \quad x=\dfrac{u}{2}-\dfrac{1}{2} \\ \\ du=2dx \end{cases} \qquad entonces
$$

$$
\begin{aligned}
\int 5x^2(2x+1)^{\frac{1}{3}}dx &= \frac{5}{2}\int x^2(2x+1)^{\frac{1}{3}}(2dx)\\
&= \frac{5}{2}\int(\frac{u}{2}-\frac{1}{2})^2u^{\frac{1}{3}}du\\
&= \frac{2}{3}\frac{1}{4}\int(\ u\ -\ 1\ )^2u^{\frac{1}{3}}du\\
&= \frac{1}{6}\int(u^2-2u+1)u^{\frac{1}{3}}du\\
&= \frac{1}{6}\int(u^{\frac{7}{3}}-2u^{\frac{4}{3}}+u^{\frac{1}{3}})du\\
&= \frac{1}{6}(\frac{3}{10}u^{\frac{10}{3}}-2\frac{3}{7}u^{\frac{7}{3}}+\frac{3}{4}u^{\frac{4}{3}})+K\\
&= \frac{1}{20}(2x+1)^{\frac{10}{3}}-\frac{1}{7}(2x+1)^{\frac{7}{3}}+\frac{1}{8}(2x+1)^{\frac{4}{3}}+K
\end{aligned}
$$

Resolver utilizandoel método de sustitución

(a) $\int x(2x-3)^{\frac{1}{3}}dx$

(b) $\int x(2x-3)^{-\frac{1}{3}}dx$

(c) $\int x^2(3x-1)^{\frac{1}{2}}dx$

(d) $\int x^2(3x-1)^{-\frac{1}{2}}dx$

(e) $\int(2x+1)^2(3x-1)^{\frac{1}{2}}dx$

(f) $\int(2x+1)^2(3x-1)^{-\frac{1}{2}}dx$

Utilizando integración por partes (integración tabular)

$$\int 5x^2(2x+1)^{\frac{1}{3}}\,dx$$

Derivar		Integrar
$p(x) = 5x^2$		$f(x) = (2x+1)^{\frac{1}{3}}$
	$\searrow$ $+$	
$10x$		$\frac{1}{2}\frac{3}{4}(2x+1)^{\frac{4}{3}}$
	$\searrow$ $-$	
10		$\frac{3}{8}\frac{1}{2}\frac{3}{7}(2x+1)^{\frac{7}{3}}$
	$\searrow$ $+$	
0		$\frac{9}{112}\frac{1}{2}\frac{3}{10}(2x+1)^{\frac{10}{3}}$

entonces

$$\int 5x^2(2x+1)^{\frac{1}{3}}\,dx = \frac{15}{8}x^2(2x+1)^{\frac{4}{3}} - \frac{90}{112}x(2x+1)^{\frac{7}{3}} + \frac{27}{224}(2x+1)^{\frac{10}{3}} + K$$

Resolver utilizando el método de integración por partes (Integración tabular)

(a) $\int x(2x-3)^{\frac{1}{3}}\,dx$

(b) $\int x(2x-3)^{-\frac{1}{3}}\,dx$

(c) $\int x^2(3x-1)^{\frac{1}{2}}\,dx$

(d) $\int x^2(3x-1)^{-\frac{1}{2}}\,dx$

(e) $\int (2x+1)^2(3x-1)^{\frac{1}{2}}\,dx$

(f) $\int (2x+1)^2(3x-1)^{-\frac{1}{2}}\,dx$