## Outline

- Relativisitic Quantum Mechanics
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#### Perturbation Theory. Interactions

Physical processes

• 
$$S \rightarrow P_1 P_2$$

## **Relativistic Quantum Mechanics**

A Short history of these concepts may be (Weinberg I)

- **1900** Planck quantization.  $E = \hbar \omega$ .  $\hbar$  rises. 'Explain' black body radiation. QFT and QM begin.
- 2 1901-3 Rutherford and Soddy radioactive decays (Rn<sup>220</sup>,  $\tau \simeq 60$  sec.) are of statistical nature:  $\tau_{1/2}$ .
- I905 Einstein again quantization (photon). Photoelectriceffect (Hertz).
- I909 Photon statistical nature. Self interference.
- 1913 Bohr: Larmor ( $P = 4\alpha a^2/3$ ) fails completely for atoms. Lifetimes  $\sim 10^{-10}$  sec.!.  $L = n\hbar$ , hydrogen spectra. Same  $\hbar$ .
- 1916 Sommerfeld elliptic orbits model. Fine structure . First attempt to unify QM with Relativity.
- **2** 1923 Compton experiment: photon scattered by electrons. Photon;  $E = \hbar \omega$  and  $\mathbf{p} = \hbar \mathbf{k}$ .

- 1926. Heisenberg, Born, Jordan, Dirac et al. Second quantization (QFT). [q, p] = iħ EM fields quantized. E and B operators. Semiclassical theory of radiation. Matter quantized but non relativistic, radiation classical.
- I 1927 Dirac was able to explain spontaneous emission using QFT.
- I928 Dirac equation . Matter quantized and relativistic, radiation classical.
- **1932** Anderson .  $e^+$  antimatter .  $\gamma \rightarrow e^-e^+$  in cosmic rays.
- 30-s. First divergences are discovered in QFT (QED) (Heisenberg, Pauli, Oppenheimer)when corrections to fine structure were attempted. Weisskopf 1934?.
- **1934.** Fermi  $\psi^4$  QFT for  $\beta$ -decay. Weak interactions.
- 1935 Yukawa first Strong Interaction theory.
- I 1934-8. W. Houston, R. Williams and S. Pastenack rumors on  $2s_{1/2} 2p_{1/2}$  nonzero energy separation.

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- **1949.** Lamb-Retherford shift.  $2s_{1/2}-2p_{1/2}$ .
- 50-s. Tomonaga, Feynman, Schwinger, Dyson, et al. Renormalizacion. 'No' divergences in QED (and in some cases QFT). Calculations: Lamb shift, anomalous magnetic moment of the electron (Schwinger). Fortunately precision experiments are possible too and the comparison theory-experiment is impressive. QED 'the best theory' ever.
- 3 1958 Yang-Mills theories . From an gauge abelian theory (U(1), QED) to nonabelian one (like SU(N)).
- 1964 Higgs spontaneous breaking symmetries. Mass: W-s, Z fermions (no nucleon).
- 50-60-s. QFT for Strong interactions?. S-matrix 'instead' QFT.

- 1963 Glaubert, coherent states (laser). Quantum Optics.
- I968 Weinberg-Salam (WSM) for the Weak (Electroweak) Interactions.
- 1973 't Hooft and Veltman WSM renormalizable .
- 1973 Gell-Mann, Leutwyler and Fritzsch QCD. Standard Model.
- 1974. Gross, Wilczek, and Politzer QCD Asymptotic Freedom.

- QFT consistent theory including Fields (like the EM ones)+Special Relativity+Quantum Mechanics . SM .
- Phenomena 'explained' (quantitative). Not predicted correctly by Classical FT:
  - How atoms radiate: Classical Larmor  $P = 4\alpha a^2/3$  cCompletely wrong for atoms. Unstable atoms. Lifetimes  $\sim 10^{-10}$  sec.
  - In Spontaneous emission in Classical Field Theory.
  - Solution Photon (electron, so on) dual character ( $E = \hbar \omega$ ,  $\mathbf{p} = \hbar \mathbf{k}$ ) needed to explain black body spectra, photoelectric effect, Compton experiment, photon self interference and so on.
  - Numer of particles is not constant. Pair creation/annihilation, like γ ↔ e<sup>-</sup>e<sup>+</sup>, μ<sup>-</sup>μ<sup>+</sup>, qq̄ (q, quark), pp̄, W<sup>-</sup>W<sup>+</sup>, etc. Similarly decay or production of particles, like a β ((Z, A) → (Z ± 1)e<sup>+</sup>ν) decay and so on.

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- Fine and hyperfine structure of atomic spectra.
- Lamb shift , now measured in many experiments: hydrogen atom and the hydrogenic atoms U<sup>92</sup>, etc.
- Spin. Statistics.
- Zeeman and Stark effects.
- The anomalous magnetic moment of the electron, muon, etc.
- In general any process involving loops like Asymptotic Freedom, the 'running' of the strong, EM constants, etc. Many processes are possible in fact at the loop level (quantum level).

Schrödinger Equation (non relativistic QM)

$$E = \frac{p^2}{m} + V(x) = H, \ \mathbf{p} \to -i\nabla, \ E \to i\frac{\partial}{\partial t}, \ [\mathbf{p}_i, \ x_j] = -i\delta_{ij}$$
$$i\frac{\partial}{\partial t}\psi = H\psi = \left[-\frac{\nabla^2}{m} + V(x)\right]\psi = \left[-\frac{\Delta}{m} + V(x)\right]\psi$$

Salpeter Equation

$$E = \sqrt{p^2 + m^2} = H$$
$$i\frac{\partial}{\partial t}\psi = H\psi = \sqrt{m^2 + \Delta}\psi$$

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#### Klein-Gordon equation. Scalar files. No spin.

$$\begin{aligned} E^2 &= p^2 + m^2 = H^2 \\ & \left(\Box + m^2\right)\phi = \mathbf{0}, \ \Box = \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta \end{aligned}$$

Hydrogenic atoms ( $V = -Z\alpha/r$ , Coulomb, classical potential)

$$E_{n, l} = m \left[ 1 + \left( \frac{Z\alpha}{n - l - 1/2 + \sqrt{(l + 1/2)^2 - (Z\alpha)^2}} \right)^2 \right]^{-1/2}$$
  

$$\simeq m \left[ 1 - \frac{1}{2} \left( \frac{Z\alpha}{n} \right)^2 - \frac{1}{2} \left( \frac{Z\alpha}{n} \right)^4 \left( \frac{n}{l + 1/2} - \frac{3}{4} \right) + \cdots \right] (1)$$

Fine structure (Sommerfeld 1916)  $l + 1/2 \neq n_{\theta} = 1, 2, 3, \cdots$ 

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#### Proca's equation (1936). Vector fields. Spin=1

$$\partial_{\mu}F^{\mu\nu} + m^{2}A^{\nu} = \Box A^{\nu} - \partial^{\nu}\partial_{\mu}A^{\mu} + m^{2}A^{\nu} = 0$$
$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$$

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# Dirac's equation (1928). Spinorial fields. Spin 1/2.

$$i\frac{\partial}{\partial t}\psi = H\psi$$
  

$$H = \alpha_{i}p_{i} + m\beta, \ H^{\dagger} = H, \ H^{2} = E^{2} = m^{2} + p^{2}$$
  

$$\alpha^{2} = \beta^{2} = 1, \ \{\alpha_{i}, \ \alpha_{j}\} = \delta_{ij}, \ \{\alpha_{i}, \ \beta\} = 0, \ (Clifford - Grassmann)$$

Covariant form

 $\sim$ 

$$[i\gamma_{\mu}\partial^{\mu}-m]\psi=0, \ \{\gamma^{\mu}, \ \gamma^{\nu}\}=2g^{\mu\nu}$$

$$\begin{split} \gamma_{0} &= \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \gamma_{i} = \beta \alpha_{i} = \gamma^{i} = \begin{pmatrix} 0 & \sigma_{i} \\ -\sigma_{i} & 0 \end{pmatrix} \\ \gamma_{5} &= -\frac{i}{4!} \epsilon_{\mu\nu\alpha\beta} \gamma^{\mu} \gamma^{\nu} \gamma^{\alpha} \gamma^{\beta} = i \gamma^{0} \gamma^{1} \gamma^{2} \gamma^{3} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma^{\mu\nu} &= \frac{i}{2} [\gamma^{\mu}, \gamma^{\nu}], \ \sigma^{0i} = i \begin{pmatrix} 0 & \sigma^{i} \\ \sigma^{i} & 0 \end{pmatrix} \quad \sigma^{ij} = \epsilon_{ijk} \begin{pmatrix} \sigma_{k} & 0 \\ 0 & \sigma_{k} \end{pmatrix} \end{split}$$

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) In this case one has to solve the equation,  $\psi \sim \exp[i p \cdot x]$ 

$$(\not p - m)\psi = \begin{pmatrix} p_0 - m & -\sigma \cdot \mathbf{p} \\ \sigma \cdot \mathbf{p} & -p_0 - m \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0$$
 (2)

to have a novanishing solution the determinant

$$\begin{vmatrix} p_0 - m & -\sigma \cdot \mathbf{p} \\ \sigma \cdot \mathbf{p} & -p_0 - m \end{vmatrix} = 0, \quad p_0^2 = m^2 + \mathbf{p}^2, \ p_0 = \pm \omega \equiv \pm \sqrt{m^2 + \mathbf{p}^2}$$

**(3)** thus  $\phi$  and for  $\chi$  become independent condition is needed

## Spin

- This is done by requiring  $\psi$  to be spin eigenfunction.
  - The spin operator is defined as  $\hat{s} = \gamma_5 s$ , with  $s^2 = s_0^2 s^2 = -1$ (So  $\hat{s}^2 = -s^2 = \lambda^2 = 1$ )
  - the spatial component points along a given direction.
  - obth operators have to commute:

$$0 = [\mathbf{p}, \mathbf{s}] = -2\gamma_5 \mathbf{s} \cdot \mathbf{p}, \ \mathbf{s} \cdot \mathbf{p} = \mathbf{s}_0 \mathbf{p}_0 - \mathbf{s} \cdot \mathbf{p} = 0$$
(4)

• A particular case is when one is interested in the helicity operator  $s = (p, E\mathbf{p}/p)/m$ 

- (the conditions  $s^2 = -1$ ,  $s \cdot p = 0$  are satisfied and  $\hat{h} = \sigma \cdot \mathbf{p} / p = \sigma \cdot \hat{\mathbf{n}}$ ).
- **6** The spin eigenvalue equation is, then  $\hat{s}\psi = \lambda\psi = \pm\psi$ :

$$(\sigma \cdot \mathbf{s} - \lambda)\phi - s_0\chi = 0$$
  

$$s_0\phi - (\sigma \cdot \mathbf{s} - \lambda)\chi = 0$$
(5)

- One can solve these equation in the CM system (with  $s_0 = 0$ ) and then boost them to the lab one, with momenta **p**.
- 2 In the CM one has two possibilities ( $s_0 = \mathbf{s} \cdot \mathbf{p} / p_0 = 0$ )

$$(p_0 - m)\phi = (p_0 + m)\chi = (\sigma \cdot \mathbf{s} - \lambda)\phi = (\sigma \cdot \mathbf{s} + \lambda)\chi = 0$$
 (6)

If  $p_0 = m$  then  $\chi = 0$  and  $\phi$  has to satisfy the eigenvalue equation:  $\sigma \cdot \mathbf{s}\phi = \lambda \phi$ .

- The two solutions are  $\phi = \chi_{1,2}(\mathbf{s})$  for  $\lambda = \pm 1$  and  $\chi = \mathbf{0}$ .
- Another two solutions are obtained for p<sub>0</sub> = -m then φ = 0 and χ has to satisfy the eigenvalue equation: σ · s<sub>χ</sub> = -λχ.
- **(b)** The two solutions are  $\chi = \chi_{1,2}(\mathbf{s})$  for  $\lambda = \pm 1$  and  $\phi = 0$ .
- **(2)** In the case of  $\mathbf{s} = s_z = (0, 0, 1)$  the solutions are

$$p_{0} = -m: u(p, s) = \begin{pmatrix} \chi_{1}(\hat{s}) \\ 0 \end{pmatrix}, u(p, -s) = \begin{pmatrix} \chi_{2}(\hat{s}) \\ 0 \end{pmatrix},$$
  
$$p_{0} = -m: v(p, s) = \begin{pmatrix} 0 \\ \chi_{2}(\hat{s}) \end{pmatrix}, v(p, -s) = \begin{pmatrix} 0 \\ \chi_{1}(\hat{s}) \end{pmatrix} (7)$$

 The final solution, once the former solutions are boosted is (see Itzykson and Greiner RQM)

$$u(p, \pm s) = \frac{\not{k} + m}{\sqrt{2m(m+E)}}u(\mathbf{k} = 0, \mathbf{s}) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \chi_{1/2}(\hat{s}) \\ \frac{\sigma \cdot \mathbf{p}}{E+m}\chi_{1/2}(\hat{s}) \end{pmatrix}$$
$$v(p, \pm s) = \frac{-\not{k} + m}{\sqrt{2m(m+E)}}v(\mathbf{k} = 0, \mathbf{s}) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \frac{\sigma \cdot \mathbf{p}}{E+m}\chi_{2/1}(\hat{s}) \\ \chi_{2/1}(\hat{s}) \end{pmatrix}$$

2 The most general solution is

$$\psi(x) = \sum_{s} \int \frac{\mathrm{d}^{3} p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E}} \left[ b(p,s) u(p,s) \mathrm{e}^{-ip \cdot x} + d^{\dagger}(p,s) v(p,s) \mathrm{e}^{ip \cdot x} \right]$$

- Antimatter, positron . Dirac, Anderson 1932.
- Hole theory, Dirac sea . Bjorken I, 64.

#### Dirac's eq. with EM

EM included as a Gauge Theory :

$$\psi \rightarrow e^{if}\psi, eA_{\mu} \rightarrow eA_{\mu} + \partial_{\mu}f$$
  
 $\partial_{\mu}\psi \rightarrow D_{\mu}\psi \equiv (\partial_{\mu} - ieA_{\mu})\psi \rightarrow e^{if}D_{\mu}\psi$ 
(10)

One has QED (EM+Relativity+QM):

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left( i D - m \right) \psi$$
(11)

- No A<sup>2</sup><sub>μ</sub> term: no photon mass.
   Interaction part is L<sub>int.</sub> = eψ̄A ψ = A<sup>μ</sup>J<sup>EM</sup><sub>μ</sub> with J<sup>EM</sup><sub>μ</sub> = eJ<sub>μ</sub> = eψ̄γ<sub>μ</sub>ψ
- 3  $\partial^{\mu} J_{\mu}^{\text{EM}} = \partial^{\mu} J_{\mu} = 0$ . Charge and probability conserved.
- Probability  $\rho = J_0 = \overline{\psi}\gamma_0\psi = |\psi|^2 \ge 0$ .



$$(i\not D - m)\psi = 0 \tag{12}$$

Nucleus much more heavier than electron. Generates static \u03c6. For electron

$$\left(i\gamma_0(\frac{\partial}{\partial t}+\boldsymbol{e}\phi)+i\gamma\cdot\nabla-\boldsymbol{m}\right)\psi=0$$
(13)

3 By multiplying by  $\gamma_0$  one obtains:

$$\begin{pmatrix} i(\frac{\partial}{\partial t} + e\phi) + i\alpha \cdot \nabla - m\beta \end{pmatrix} \psi = 0 H\psi = i\partial\psi/\partial t, \ H = m\beta + \alpha \cdot \mathbf{p} - e\phi = m\beta + \alpha \cdot \mathbf{p} + V(\mathbf{r}) = m\beta + K + V H = \begin{pmatrix} m - e\phi & \sigma \cdot \mathbf{p} \\ \sigma \cdot \mathbf{p} & -m - e\phi \end{pmatrix}$$
(14)

the Schrödinger form. Where  $m\beta$  is the rest energy,  $K = \alpha \cdot \mathbf{p}$  is the 'kinetic energy,  $V(\mathbf{r}) = -e\phi(\mathbf{r})$  is potential energy.

Thus *H* is time independent and

$$\psi = e^{-iEt} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$
(15)

the equation becomes

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#### **Central potentials**

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• For central potentials the angular part becomes universal and one can use the identities ( $\mathbf{L} = \mathbf{r} \land \mathbf{p}$ 

$$\sigma \cdot \mathbf{p} = \frac{1}{r^2} (\sigma \cdot \mathbf{r})^2 \sigma \cdot \mathbf{p} = \sigma \cdot \hat{\mathbf{r}} \left( \hat{\mathbf{r}} \cdot \mathbf{p} + \frac{i\sigma \cdot \mathbf{L}}{r} \right), \ \mathbf{J} = \mathbf{L} + \mathbf{s} = \mathbf{L} + \frac{\sigma}{2}$$

Adding the two angular momenta

$$\sigma \cdot \mathbf{L} = J^2 - L^2 - 3/4, \ Y_l^{Jm} = \sum_{m_l, m_s} C_{lm_l, 1/2m_s}^{Jm} Y^{lm_l} \chi_{1/2, m_s}$$

$$\sigma \cdot \mathbf{L} Y^{Jm}_{l=J+1/2} = -(J+3/2) Y^{Jm}_{l=J+1/2}, \ \sigma \cdot \mathbf{L} Y^{Jm}_{l=J-1/2} = (J-1/2) Y^{Jm}_{l=J+1/2}$$

- 3 The orbital angular momenta **L** is not conserved,  $[H, L] \neq 0$ .
- Total angular momenta is conserved ([H, J] = 0).

#### EM interactions conserve parity P and that, for spinors

$$P\psi(t, \mathbf{x}) = \beta\psi(t, -\mathbf{x}), \ PY^{lm_l} = (-1)^l Y^{lm_l}$$
(18)

**2** Then parity can be  $(-1)^{J-1/2}$  or  $(-1)^{J+1/2}$ , respectively:

$$\psi_{Jm}^{(1)} = \begin{pmatrix} F(r)Y_{Jm}^{Jm} - 1/2 \\ if(r)Y_{lm}^{Jm} - 1/2 \end{pmatrix} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \quad \psi_{Jm}^{(2)} = \begin{pmatrix} G(r)Y_{Jm}^{Jm} - 1/2 \\ ig(r)Y_{lm}^{Jm} - 1/2 \end{pmatrix} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$
(19)

J, m and parity are well defined. / is not well defined.
Now the Dirac equation becomes

$$(E + e\phi - \mu)F(r)Y_{J-1/2}^{Jm} + \sigma \cdot \hat{\mathbf{r}} \left( \hat{\mathbf{r}} \cdot \mathbf{p} + \frac{i\sigma \cdot \mathbf{L}}{r} \right) if(r)Y_{J+1/2}^{Jm} = 0$$
  
$$(E + e\phi + \mu)if(r)Y_{J+1/2}^{Jm} + \sigma \cdot \hat{\mathbf{r}} \left( \hat{\mathbf{r}} \cdot \mathbf{p} + \frac{i\sigma \cdot \mathbf{L}}{r} \right) F(r)Y_{J-1/2}^{Jm} = 0$$
(20)

Solution Given that  $\sigma \cdot \hat{\mathbf{r}} Y_{l=J\pm 1/2}^{Jm} = -Y_{l=J\mp 1/2}^{Jm}$ .  $\kappa = J + 1/2$ .

$$(E + e\phi - \mu)F(r) - \left(\frac{d}{dr} + \frac{\kappa + 1}{r}\right)f(r) = 0$$
  
$$(E + e\phi + \mu)f(r) + \left(\frac{d}{dr} - \frac{\kappa - 1}{r}\right)F(r) = 0$$
 (21)

## Spherical waves

In the case of free particle or spherical waves ( $\phi = 0$ )

$$(E - \mu)F(r) - \left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa + 1}{r}\right)f(r) = 0$$
  

$$(E + \mu)f(r) + \left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa - 1}{r}\right)F(r) = 0$$
  

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{2}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + 1 + \frac{\kappa(\kappa - 1)}{\rho^2}\right]F = 0$$
(22)

with  $\rho = \sqrt{E^2 - \mu^2} r$  ( $E > \mu$ ). The regular solution

$$F(\rho) = Aj_{\kappa-1}(\rho) = Aj_{J-1/2}(\sqrt{E^2 - \mu^2} r)$$
(23)

so, using the recurrence relations for the spherical Bessel functions

$$\psi_{Jm} = A \left( \begin{array}{c} j_{J-1/2} (\sqrt{E^2 - \mu^2} r) Y_{J-1/2}^{Jm} \\ [i/(2J+1)] j_{J+1/2} (\sqrt{E^2 - \mu^2} r) Y_{J+1/2}^{Jm} \end{array} \right)$$
(24)

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## Hydrogenic atoms

$$\left(E - \mu + \frac{Z\alpha}{r}\right)F - \left(\frac{d}{dr} + \frac{\kappa + 1}{r}\right)f = 0, \ e\phi = \frac{Z\alpha}{r}$$
$$\left(E + \mu + \frac{Z\alpha}{r}\right)f + \left(\frac{d}{dr} - \frac{\kappa - 1}{r}\right)F = 0$$
(25)

taking  $f = u_1/r$  and  $F = u_2/r$ 

$$ru'_{1} = -\kappa u_{1} + (Z\alpha + (E - \mu)r) u_{2}$$
  

$$ru'_{2} = -(\kappa + (E + \mu)r) u_{1} + \kappa u_{2}$$
(26)

 $\lambda = \sqrt{\mu^2 - E^2}, u_1 = \sqrt{\mu - E} \cdot \exp(-\lambda r)h, u_2 = \sqrt{\mu + E} \cdot \exp(-\lambda r)H$ 

$$rh' = (\lambda r - \kappa)h + \left(-\lambda r + Z\alpha\sqrt{\frac{\mu + E}{\mu - E}}\right)H$$
$$rH' = -\left(\lambda r + Z\alpha\sqrt{\frac{\mu - E}{\mu + E}}\right)h + (\kappa + \lambda r)H$$

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and

$$r(h' + H') = \frac{Z\alpha E}{\lambda}(h + H) - \left(\kappa + \frac{Z\alpha\mu}{\lambda}\right)(h - H)$$
$$r(h' - H') = \left(\frac{Z\alpha\mu}{\lambda} - \kappa\right)(h + H) + \left(2\lambda r - \frac{Z\alpha E}{\lambda}\right)(h - H)$$
(28)

taking  $\varphi_1 = h + H = \rho^{\gamma} F_1$  and  $\varphi_2 = h - H = \rho^{\gamma} F_2$  and  $\rho = 2\lambda r$ 

$$\rho \dot{F}_{1} = \left(\frac{Z\alpha E}{\lambda} - \gamma\right) F_{1} - \left(\kappa + \frac{Z\alpha \mu}{\lambda}\right) F_{2}$$
$$\rho \dot{F}_{2} = \left(\frac{Z\alpha \mu}{\lambda} - \kappa\right) F_{1} + \left(\rho - \frac{Z\alpha E}{\lambda} - \gamma\right) F_{2}$$
(29)

with  $\gamma^2 = k^2 - \alpha^2$ ,  $a = \gamma - Z\alpha/\lambda$ ,  $b = 2\gamma + 1$ . Eliminating  $F_2$ 

$$\rho \ddot{F}_1 + (b - \rho) \dot{F}_1 - aF_1 = 0 \tag{30}$$

- The regular solution is the confluent hypergeometric function  $F_1 = AF(a, b, \rho)$ .
- **2** Given its behavior at  $\rho \to \infty$ ,  $F_1 \to \exp[\rho]$ , so  $\psi \to \exp[\rho/2]$ .
- Solving for *E* ( $\kappa = J + 1/2$ ) The function has to be a polynomial and  $a = -n_r = 0, 1, 2, \cdots$ .

$$E_{n, J} = \mu \left[ 1 + \left( \frac{Z\alpha}{n_r + \gamma} \right)^2 \right]^{-1/2} = \mu \left[ 1 + \left( \frac{Z\alpha}{n - \kappa + \sqrt{\kappa^2 - (Z\alpha)^2}} \right)^2 \right]^{-1/2}$$
$$\simeq \mu - \frac{\mu}{2} \left( \frac{Z\alpha}{n} \right)^2 - \frac{\mu}{2} \left( \frac{Z\alpha}{n} \right)^4 \left[ \frac{n}{J + 1/2} - \frac{3}{4} \right] + \cdots$$

The rest mass, the Bohr's nonrelativistic spectra, and the fine structure (Sommerfeld).









Hydrogen fine structure and hyperfine structure for the n=3 + 2 transition. (After Ohanian, Modern Physics, Ch 7., spectrum from T. W. Hansch, Stanford Univ.)

El. $(2S_{1/2} - 1S_{1/2})$	$\nu_{\mathrm{exp.}}$ [Mhz]	$\nu_{\rm theo.}$ [Mhz]
$\nu_H$	2466 061 413.187 035 (10)	2466 061 413.187 103(46)
$\nu_D - \nu_H$	670 994.334 64 (15)	670 999.586 6(15)(15)*
$ u_{\mu^+ e^-}$	2455 528 941.0(98)	2455 528 935.4(14)

Table : 2S - 1S transitions. \*  $r_d^2 - r_p^2 = 3.8212(15) \text{ fm}^2$ 

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#### QFT (canonical quantization)



 $[x_i, p_i] = i\delta_{ij}\hbar$ 

$$p_i = \frac{\delta \mathcal{L}}{\delta x_i} \rightarrow -i \frac{\partial}{\partial q_i}$$



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## Canonical quantization example: Scalar Field

Lagrangian (Normal product understood everywhere. Defined later)

$$\mathcal{L} = (\partial_{\mu}\phi)^{\dagger}(\partial^{\mu}\phi) - m^{2}\phi^{\dagger}\phi$$
(32)

Provide a conjugate momenta are

$$\pi = \frac{\delta \mathcal{L}}{\delta \partial_0 \phi} = \partial_0 \phi^{\dagger}, \ \pi^{\dagger} = \frac{\delta \mathcal{L}}{\delta \partial_0 \phi^{\dagger}} = \partial_0 \phi$$
(33)

Ommutation relations are, at equal times

$$[\pi(x), \ \phi(y)] = -i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \ [\pi^{\dagger}(x), \ \phi^{\dagger}(y)] = -i\delta^{(3)}(\mathbf{x} - \mathbf{y})$$
(34)

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$$\phi(\mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2} \sqrt{2\omega}} \left[ \mathbf{a}(\mathbf{k}) \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} + \mathbf{b}^{\dagger}(\mathbf{k}) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \right]$$
(35)

with  $[a(k), a^{\dagger}(k')] = [b(k), b^{\dagger}(k')] = \delta^{(3)}(\mathbf{k} - \mathbf{k}')$  and  $k_0 = \omega = \sqrt{m^2 + \mathbf{k}^2}$ . Otherwise they commute.

Pock space: particle (antiparticle) states:

$$|p>\equiv a^{\dagger}(p)|0>, \ |ar{p}>\equiv b^{\dagger}(p)|0>, \ a|0>=b|0>=0$$
 (36)

The 'creation'  $(a^{\dagger})$  and 'annihilation' (a).

Solution Normalized  $= \delta^{(3)}(\mathbf{p} - \mathbf{p}')$ 

$$|n_{1}(p_{1})\cdots n_{l}(p_{l})\rangle = \frac{1}{\sqrt{n_{1}!\cdots n_{l}!}} (a^{\dagger}(p_{1}))^{n_{1}}\cdots (a^{\dagger}(p_{l}))^{n_{l}} |0\rangle (37)$$

Oumber operator

$$N = \int \mathrm{d}^3 k \ a^{\dagger}(k) a(k), \ \bar{N} = \int \mathrm{d}^3 k \ b^{\dagger}(k) b(k) \qquad (38)$$

Infinite to finite volume (Bjorken II 26)

$$\int \mathrm{d}^{3}k \to \sum_{k} \Delta V_{k}, \ \boldsymbol{a}(k) \to \frac{\boldsymbol{a}_{k}}{\sqrt{\Delta V_{k}}}, \ \delta^{(3)}(\mathbf{k} - \mathbf{k}') \to \frac{\delta_{kk'}}{\Delta V_{k}}$$
(39)

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The energy-momentum tensor and the energy-momentum of a particle

$$\begin{split} \mathcal{T}^{\mu\nu} &= -g^{\mu\nu}\mathcal{L} + \frac{\delta\mathcal{L}}{\delta\partial_{\mu}\phi}\partial_{\nu}\phi \\ &= -g^{\mu\nu}\left[(\partial_{\alpha}\phi)^{\dagger}(\partial^{\alpha}\phi) - m^{2}\phi^{\dagger}\phi\right] + (\partial^{\mu}\phi)^{\dagger}(\partial^{\nu}\phi) + (\partial^{\mu}\phi)(\partial^{\nu}\phi)^{\dagger} \\ \int d^{3}x \,\phi^{\dagger}\phi &= \int \frac{d^{3}k}{2\omega} \left[a^{\dagger}(k)a(k) + b(k)b^{\dagger}(k) + a^{\dagger}(\mathbf{k})b^{\dagger}(-\mathbf{k}) \cdot e^{2i\omega t} + b(\mathbf{k})a(-\mathbf{k}) \cdot e^{-2i\omega t}\right] \\ \int d^{3}x \,\partial^{\mu}\phi^{\dagger}\partial^{\nu}\phi &= \int \frac{d^{3}k}{2\omega} \left[\left\{a^{\dagger}(k)a(k) + b(k)b^{\dagger}(k)\right\}k^{\mu}k^{\nu} \\ &+ \left\{a^{\dagger}(\mathbf{k})b^{\dagger}(-\mathbf{k}) \cdot e^{2i\omega t} - b(\mathbf{k})a(-\mathbf{k}) \cdot e^{-2i\omega t}\right\}k^{\mu}k^{\nu}\right] \\ \hat{\rho}^{\mu} &= \int d^{3}x \,\mathcal{T}^{\mu0} = \int d^{3}k \,\left[a^{\dagger}(k)a(k) + b^{\dagger}(k)b(k)\right]k^{\mu} \\ p^{\mu} &\equiv \frac{\langle p|\hat{p}^{\mu}|p \rangle}{\langle p|p \rangle} = p^{\mu} \end{split}$$
(40)

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- In particular  $p_0 = E = \omega > 0$ . Similarly for the antiparticle!
- Zero point energy: If no Normal product is considered

$$\boldsymbol{\rho}^{\mu} \equiv \langle 0|\hat{\boldsymbol{\rho}}^{\mu}|0\rangle = \infty \tag{41}$$

This energy is subtracted by taking Normal products everywhere

- Zero Point energy (Planck, Einstein, Stern 1913). Seen in Casimir effect (Itzykson 138). Important in nanotechnology.
- Spontaneous photon emissions.
- Normal and Time ordered product for bosons, and fermions respectively

$$N(a^{\dagger}a) = \pm N(aa^{\dagger}) = a^{\dagger}a$$
  

$$T(\phi(x)\phi(y)) = \phi(x)\phi(y)\theta(t_x - t_y) \pm \phi(y)\phi(x)\theta(t_y - t_x)$$
(42)

The electric current and charge are

$$J^{\mu} = iN\left(\phi^{\dagger}\partial^{\mu}\phi - \phi\partial^{\mu}\phi^{\dagger}\right), \ \hat{Q} = \int d^{3}x \ J^{0}$$
$$q = \langle p|\hat{Q}|p \rangle = i\left(-\frac{ip_{0}}{2\omega} - \frac{ip_{0}}{2\omega}\right) = 1$$
$$\bar{q} = \langle \bar{p}|\hat{Q}|\bar{p}\rangle = i\left(\frac{ip_{0}}{2\omega} - \frac{-ip_{0}}{2\omega}\right) = -1$$
(43)

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#### The propagator is defined

$$<0|T(\phi^{\dagger}(x)\phi(y))|0> = \int \frac{d^{3}k \, d^{3}k'}{(2\pi)^{3}\sqrt{2\omega \cdot 2\omega'}} < 0|T\left(a^{\dagger}e^{ik\cdot x} + be^{-ik\cdot x}\right)\left(a'e^{-ik'\cdot y} + b'^{\dagger}e^{ik'\cdot y}\right)|0> \\ = \int \frac{d^{3}k \, d^{3}k'}{(2\pi)^{3}\sqrt{2\omega \cdot 2\omega'}} < 0|a'a^{\dagger}e^{i(k\cdot x - k'\cdot y)}\theta(y - x) + bb'^{\dagger}e^{-i(k\cdot x - k'\cdot y)}\theta(x - y)|0> \\ = \int \frac{d^{3}k \, d^{3}k'}{(2\pi)^{3}\sqrt{2\omega \cdot 2\omega'}} \delta^{(3)}(\mathbf{k} - \mathbf{k}')\left[e^{i(k\cdot x - k'\cdot y)}\theta(y - x) + e^{-i(k\cdot x - k'\cdot y)}\theta(x - y)\right] \\ = \int \frac{d^{3}k}{(2\pi)^{3}2\omega}\left[e^{ik\cdot (x - y)}\theta(y - x) + e^{-ik\cdot (x - y)}\theta(x - y) = i\Delta_{F}(x - y)\right] \\ i\Delta_{F}(x - y) = i\int \frac{d^{4}k}{(2\pi)^{2}}\frac{\exp[-ik\cdot (x - y)]}{k^{2} - m^{2} + i\epsilon} \\ \left(\Box + m^{2}\right)\Delta_{F}(x - y) = -\delta^{(4)}(x - y)$$
(44)

## Perturbation Theory. Interactions

$$\begin{aligned} |\psi(t)\rangle_{I} &= U(t,t_{0})|\psi(t_{0})\rangle_{I}, \ i\frac{\partial}{\partial t}U(t,t_{0}) = H_{I}U(t,t_{0}), \ H = H_{0} + H_{I} \\ U(t,t_{0}) &= T\exp\left[-i\int_{t_{0}}^{t}dt'H_{I}(t')\right], \\ S &\equiv \lim_{t,t_{0}\to\pm\infty}U(t,t_{0}) = T\exp\left[\int d^{4}x\mathcal{L}_{I}(x)\right] \end{aligned}$$
(45)

$$<0|T(\phi(x_1)\cdots\phi(x_n))|0>=\frac{<0|T(\phi(x_1)\cdots\phi(x_n)\exp[i\int \mathrm{d}x\ \mathcal{L}_l])|0>}{<0|T(\exp[i\int \mathrm{d}x\ \mathcal{L}_l])|0>=}$$
(46)

The amplitude is the S-matrix normalized ( $= \delta^{(3)}(\mathbf{p} - \mathbf{p}') = V/(2\pi)^3$ )

$$A = \frac{\langle f|S|i\rangle}{\sqrt{\langle f|f\rangle\langle i|i\rangle}}$$
(47)

Thus the differential partial width is, for a particle decaying to *n* new particles

$$d\Gamma = \frac{|A|^2}{T} \frac{V d^3 p_1}{(2\pi)^3} \cdots \frac{V d^3 p_n}{(2\pi)^3}$$
(48)

For collisions the flux, F of the incident particles is

$$F = \frac{1}{V} \frac{|\langle i|\mathbf{J}|i\rangle|}{|\langle i|i\rangle|} = \frac{1}{V} \left| \frac{\mathbf{p}}{E} - \frac{\mathbf{p}'}{E'} \right|$$
(49)

and the differential cross section is

$$d\sigma = \frac{1}{F} \frac{|A|^2}{T} \frac{V d^3 p_1}{(2\pi)^3} \cdots \frac{V d^3 p_n}{(2\pi)^3}$$
(50)

The Lagrangian is  $\mathcal{L} = g_S m_S S P_1 P_2$ , where *S* and *P* are scalar meson. The *S*-matrix, the Amplitude and the matrix  $\mathcal{M}$  are, at tree level are

$$S \simeq i \int dx < S |\mathcal{L}| P_1 P_2 >= ig_S m_S \frac{(2\pi)^4 \delta^{(4)} (p - k_1 - k_2)}{(2\pi)^{9/2} \sqrt{8\omega_S \omega_1 \omega_2}}$$
  
$$A \simeq ig_S m_S \frac{(2\pi)^4 \delta^{(4)} (p - k_1 - k_2)}{V^{3/2} \sqrt{8\omega_S \omega_1 \omega_2}} \equiv \mathcal{M} \frac{(2\pi)^4 \delta^{(4)} (p - k_1 - k_2)}{V^{3/2} \sqrt{8\omega_S \omega_1 \omega_2}} (51)$$

where the states are normalized as  $\langle i|i \rangle = V/(2\pi)^3$ ,  $\langle f|f \rangle = V^2/(2\pi)^6$ ,  $\mathcal{M} \equiv ig_S m_S = \langle S|P_1P_2 \rangle$  and  $\omega_i = \sqrt{m_i^2 + \mathbf{k}_i^2}$ . Then

$$d\Gamma = \frac{1}{2\omega_{S}} |\mathcal{M}|^{2} (2\pi)^{4} \delta^{(4)} (p - k_{1} - k_{2}) \frac{\mathrm{d}^{3} k_{1}}{2\omega_{1} (2\pi)^{3}} \frac{\mathrm{d}^{3} k_{2}}{2\omega_{2} (2\pi)^{3}}$$
(52)

At the *S* rest frame  $\mathbf{k} \equiv \mathbf{k}_1 = -\mathbf{k}_2$  and  $m_S = \omega_1 + \omega_2$ , so  $k \equiv |\mathbf{k}| = \lambda^{1/2} (m_S^2, m_1^2, m_2^2)/2m_S \rightarrow (m_S/2)\sqrt{1 - 4m_P^2/m_S^2}$ 

$$\begin{split} \Gamma(S \to P_1 P_2) &= \frac{1}{8} g_S^2 m_S \int \frac{\mathrm{d}^3 k}{(2\pi)^2 4 \omega_1 \omega_2} \delta(m_S - \omega_1 - \omega_2) \\ &= \frac{g_S^2}{8\pi m_S} \lambda^{1/2}(m_S^2, m_1^2, m_2^2) \to \frac{g_S^2 m_S}{16\pi} \sqrt{1 - 4m_\pi^2 / m_S^2} 3) \end{split}$$

given that  $d^3k = 4\pi k^2 dk$  and  $\int \delta(f(x)) dx = \sum_i |f'(x_i)|^{-1}$  with  $f(x_i) = 0$ . In the case of two identical final scalars ( $P_1 = P_2 = P$ ) a factor of 1/2! has to be added

$$\Gamma = \frac{g_S^2}{16\pi} \frac{m_S}{2} \sqrt{1 - 4m_P^2/m_S^2}$$
(54)

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$$\Gamma(H \to WW) = \frac{g^2}{64\pi} \frac{m_H^3}{m_W^2} \left(1 - x_W + \frac{3}{4} x_W^2\right) \sqrt{1 - x_W}$$
  

$$\Gamma(H \to ZZ) = \frac{g^2}{128\pi} \frac{m_H^3}{m_W^2} \left(1 - x_Z + \frac{3}{4} x_Z^2\right) \sqrt{1 - x_Z} \simeq \frac{1}{2} \Gamma(H \to WW)$$
  

$$\Gamma(H \to f\bar{f}) = \frac{N_C G_F}{4\sqrt{2}\pi} m_H m_f^2 \beta_f^3 \left(1 + \frac{4\alpha_S}{3\pi} \left(\frac{9}{4} + \frac{3}{2} \log(m_q^2/m_H^2)\right)\right) (55)$$

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The transition matrix is

$$\mathcal{M} = i \frac{q_i q_f e^2}{q^2} \bar{u}(3) \gamma_\mu v(4) \bar{v}(2) \gamma^\mu u(1)$$
(56)

Computing the traces

$$|\mathcal{M}|^2 = \frac{q_i^2 q_f^2 e^4}{q^4} \frac{2}{(mM)^2} \left[ (p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2 + (m^2 + M^2)(m^2 + p_1 \cdot p_4)^2 \right]$$

Working out the kinematics

$$(p_{1} \cdot p_{3})^{2} + (p_{1} \cdot p_{4})^{2} = 2E^{4}[1 + (\beta\beta'x)^{2}]$$
  

$$(p_{1} \cdot p_{3})^{2} - (p_{1} \cdot p_{4})^{2} = -4E^{4}\beta\beta'x$$
  

$$m^{2} + p_{1} \cdot p_{2} = M^{2} + p_{3} \cdot p_{4} = 2E^{2} = s/2$$
(58)

with  $s = q^2 = 4E^2$ ,  $x = \cos \theta$  where  $\theta$  is the angle between  $\mathbf{p}_1$  and  $\mathbf{p}_{3_{2,0,0}}$ 

Adding all terms and replacing in the formula

$$\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega} = \frac{1}{32\pi} \left(\frac{mM}{E}\right)^2 |\mathcal{M}|^2 \beta' \tag{59}$$

one obtains

$$\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}x} = (q_i q_f)^2 \beta' \frac{\pi \alpha^2}{2s} \left[ 1 + (\beta \beta' x)^2 + \frac{m^2 + M^2}{E^2} \right]$$
(60)

Particular cases can be obtained for the high energy limit (E >> m, M)

$$\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}x} = (q_i q_f)^2 \frac{\pi \alpha^2}{2s} \tag{61}$$

and

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$$\bar{\sigma} = (q_i q_f)^2 \frac{4\pi\alpha^2}{3s} \rightarrow \bar{\sigma}_{\text{QED}}(ee^+ \rightarrow \mu\mu^+) = \frac{4\pi\alpha^2}{3s}$$
(62)

$$\bar{\sigma} = (q_i q_f)^2 \frac{4\pi \alpha^2}{3s} [1 + 2v_f v_i + (v_f^2 + a_f^2)(v_i^2 + a_i^2)] \to \bar{\sigma}_{\text{QED}}(ee^+ \to \mu\mu^+)$$

$$R = \frac{\bar{\sigma}(e^-e^+ \to \text{hadrons})}{\sigma_{\text{QED}}(ee^+ \to \mu\mu^+)} \simeq N_C \sum_f (q_f)^2 \simeq \begin{cases} 2 & \text{u, d, s} \\ 10/3 & \text{u, d, s, c} \\ 11/3, & \text{u, d, s, c, b.} \end{cases}$$

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SM windows to NP

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Figure 30, Figure 337. Wold have the hold can see that  $d^{-1} = -i \operatorname{holders} and have that <math>d^{-1} = e^{i} f^{-1} = -i \operatorname{holders} (f^{-1} + e^{i}) = \int_{-\infty}^{\infty} -i \operatorname{holders} (f^{-1} + e^{i}) = \int_$ 

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Exper.	$\alpha^{-1}$	acc.
CODATA 2010	137.035 999 074(44)	
<i>g</i> <sub>e</sub> – 2	137.035 999 174(35)	
$g_{\mu}$ $-$ 2	137.035 999 710(96)?	
$R_{\infty}^{},^{86}$ Rb [ <b>?</b> ], $^{133}$ Cs	137.035 999 037(91)	
<i>h/m<sub>n</sub>d</i> <sub>220</sub> , Si	137.036 007 7(28)	
$R_{\infty}, \lambda_n^{\text{Compton}}$	137.036 010 1(54)	
Hyperfine	137.036 0(3)	
Hyperfine muonium	137.036 001 7(80)	
Lamb shift	137.036 8(7)	
Q Hall e. $R_K = h/e^2$	137.036 003 9(25)	
Josephson e. $K_J = 2e/h$	137.035 987 5(43)	

Table : Experimental  $\alpha$ -s, from CODATA-NIST at [?, ?]

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Element	$\nu_{\rm exp.}$ [Mhz]	$\nu_{\mathrm{theo.}}$ [Mhz]
Н	1057.8446 (29)	1057.851 (2)
D	1059.2337 (29)	1059.271 (25)
He <sup>+</sup>	14041.13 (17)	14041.18 (13)
C <sup>5+</sup>	780.1 (80) GHz	782.87 GHz
phosphorus	20 188 (29) GHz	20 254(10) GHz
Mg <sup>11+</sup> $\lambda_{1s_{1/2}-2p_{1/2}}$	0.842 50(4) nm	?? GHz
Mg <sup>11+</sup> $\lambda_{1s_{1/2}-2p_{3/2}}$	0.841 90(2) nm	?? GHz
U <sup>91+</sup>	460.2 (46) eV	463.95 (50) eV
$H \nu_L(1S)$	8172.837 (22)	8172.731 (40)
$D \nu_L(1S)$	8183.966 (22)	8172.811 (32)?

Table : Lamb shift for hydrogen and other elements (Izk. 365, Jauch 534) [?]. See Phys. News. update # 242, D. Berkenland in ref [?].

Element	$\nu_{\rm exp.}$ [Mhz]	$\nu_{\rm theo.}$ [Mhz]
H, 1s	1420.405 751 766 7(10)	1420.452
D, 1s	327.384 352 522 (2)	327.339
Tritium, 1s	1 516. 701 470 773 (8)	1516.760
<sup>3</sup> He <sup>+</sup> , 1s	8665.649 867 (10)	8667.494
μe	4463.302 776 (51)	4463.302 913 (511)(34)(220)
e^-e^+	203 388.65 (67)	203 391.69 (41)
H, 2s	177.556 838 (4)	177.556 8381(4)
D, 2s	40.924 454 (7)	?40.918 81
<sup>3</sup> He <sup>+</sup> , 2s	1083.354 980 7 (88)	?1083.5853

Table : Hyperfine splitting for hydrogen and other elements. [?]

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$$\begin{aligned} a_{\theta}^{\text{QED}} &= \frac{\alpha}{2\pi} - 0.328\ 478\ 445 \cdot \left(\frac{\alpha}{\pi}\right)^2 + 1.183(11) \cdot \left(\frac{\alpha}{\pi}\right)^3, \\ a_{\mu}^{\text{SM}} &= \frac{\alpha}{2\pi} + 0.765\ 782 \cdot \left(\frac{\alpha}{\pi}\right)^2 + 24.45(6) \cdot \left(\frac{\alpha}{\pi}\right)^3 \\ a_{l}^{\text{QED}} &= \frac{\alpha}{2\pi} + \left[\frac{197}{144} + \frac{\pi^2}{12} + \frac{3}{4}\zeta(3) - \frac{\pi^2}{2}\ln 2\right] \left(\frac{\alpha}{\pi}\right)^2 + \cdots \\ a_{\tau}^{\text{EW}}(1\ \text{loop}) &= \frac{5G_{\mu}m_{\tau}^2}{24\sqrt{2}\ \pi^2} \left[1 + \frac{1}{5}\left(1 - 4s_W^2\right)^2 + \mathcal{O}\left(m_{\tau}^2/m_{Z,\ W,\ H}\right)\right] \\ a_{\tau}^{(2)}(\text{Had.}) &= \int_{4m_{\pi}^2}^{\infty} \mathrm{d}s\ K_{\tau}(s)R(s),\ K(s) = \frac{\alpha^2}{3\pi^2s} \int_0^1 \mathrm{d}s \frac{x^2(1-x)}{x^2 + (1-x)s/m} \\ R(s) &= \frac{\sigma^{(0)}(e^-e^+ \to \text{had.})}{\sigma^{(0)}(e^-e^+ \to \mu^-\mu^+)} \end{aligned}$$
(6)

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A four-vector  $A^{\mu} = (A^0, \mathbf{A})$  transforms under the Lorentz transformations (with v the relative velocity along the *x*-axis, and in natural units c = 1)

$$A^{0} = \gamma (A'^{0} + vA'^{1}) = A'^{0} \cosh \xi + A'^{1} \sinh \xi, \ A^{2} = A'^{2}, \ A^{3} = A'^{3}$$
  

$$A^{1} = \gamma (A'^{1} + vA'^{0}) = A'^{1} \cosh \xi + A'^{0} \sinh \xi$$
  

$$\mathbf{x} = \mathbf{x}' + \left[\frac{\mathbf{x}' \cdot \mathbf{v}}{v^{2}}(\gamma - 1) + \gamma t'\right] \mathbf{v}, \ t = \gamma (t' + \mathbf{v} \cdot \mathbf{x}')$$
(65)

with  $\gamma = 1/\sqrt{1 - v^2}$ ,  $\cosh \xi = \gamma$ ,  $\sinh \xi = \gamma v$  (so  $\tanh \xi = v \le 1$ ) and given that  $\cosh^2 \xi - \sinh^2 \xi = 1$ . In particular the norm, and the inner (or dot) product between two four-vectors are invariant.

$$|A|^2 = (A^0)^2 - A^2 = A^\mu g_{\mu\nu} A^\nu = A^\mu A_\mu, \ A_\mu \equiv g_{\mu\nu} A^\nu = (A^0, -A^0_\mu)$$

is invariant with ( $A^{\mu}$  and  $A_{\mu}$  are the covariant and contravariant components)

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$$g_{\mu\nu} = \text{diag.}(1, -1, -1, -1), \ g_{\mu\nu}g^{\nu\mu'} \equiv g_{\mu}^{\ \mu'} = \delta_{\mu}^{\ \mu'}, \ g^{\mu\nu} = g_{\mu\nu}$$
 (67)

Examples of four-vectors are

$$\begin{aligned} x^{\mu} &= (t, \mathbf{x}), \ s^{2} = t^{2} - (\mathbf{x})^{2}, \ \mathbf{u}^{\mu} \equiv \frac{d\mathbf{x}^{\mu}}{d\tau} = \gamma_{\mathbf{u}}(\mathbf{1}, \mathbf{u}), \ \mathbf{a}^{\mu} \equiv \frac{d^{2}\mathbf{x}^{\mu}}{d^{2}\tau} = \gamma_{\mathbf{u}}^{4} \left( p^{\mu} = mv^{\mu} = m\gamma(\mathbf{1}, \mathbf{v}) = (E, \mathbf{p}) = i\partial^{\mu} = i\frac{\partial}{\partial x_{\mu}} = i\left(\frac{\partial}{\partial t}, -\nabla\right), \ \partial_{\mu} \equiv \partial^{\mu}x_{\nu} &= g^{\mu}_{\nu} = \frac{\partial x_{\nu}}{\partial x_{\mu}} = \delta_{\mu\nu}, \ F^{\mu} = ma^{\mu} = \gamma_{u}(\mathbf{F} \cdot \mathbf{u}, \mathbf{F}) = \gamma_{u}(P, \mathbf{F}), \ k^{\mu} = (a^{\mu}\mathbf{F}) = \frac{d\mathbf{p}}{dt} = m\gamma_{u}^{3}(\mathbf{a} \cdot \mathbf{u})\mathbf{u} + m\gamma_{u}\mathbf{a} = m\gamma_{u}^{3}\mathbf{a}_{\parallel} + m\gamma_{u}\mathbf{a}_{\perp}, \ \mathbf{a}_{\parallel} = \frac{\mathbf{a} \cdot \mathbf{u}}{u^{2}}\mathbf{u}, \ \mathbf{a}_{\perp} = J^{\mu} = (\rho, \mathbf{J}), \ A^{\mu} = (\phi, \mathbf{A}), \ \gamma^{\mu} = (\gamma^{0}, \gamma^{i}) \end{aligned}$$

with  $dt = \gamma d\tau$ , been  $\tau$  the proper time.

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Several norms are  $p^{\mu}p_{\mu} = E^2 - p^2 = m^2 v^{\mu}v_{\mu} = 1$ . Velocities are added as

$$u_{x} = \frac{u'_{x} + v}{1 + u'_{x}v}, \ u_{y} = \frac{u'_{y}}{\gamma(1 + u'_{x}v)}, \ \gamma_{u} = \gamma\gamma_{u'}(1 + u'_{x}v)$$
(69)

In general Lorentz transformations can be written as  $x^{\mu} = \Lambda^{\mu}_{\nu} x'^{\nu}$ . In the special case of *v* going along the x-axis

$$\Lambda^{\mu}_{\ \nu} = \begin{pmatrix} \gamma & \gamma \nu & 0 & 0 \\ \gamma \nu & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \Lambda_{\mu\nu} = g_{\mu\mu'} \Lambda^{\mu'} \nu, \cdots$$
(70)

Given that the norm has to be invariant one has that

$$g_{\mu\nu}x^{\mu}x^{\nu} = g_{\mu\nu}\left(\Lambda^{\mu}_{\mu'}x'^{\mu'}\right)\left(\Lambda^{\nu}_{\nu'}x'^{\nu'}\right) = g_{\mu\nu}x'^{\mu}x'^{\nu}$$
(71)

and  $\Lambda^{\mu}_{\ \alpha}g_{\mu\nu}\Lambda^{\nu}_{\ \beta} = \Lambda^{\mu}_{\ \alpha}\Lambda_{\mu\beta} = (\Lambda^{T})^{\ \mu}_{\ \alpha}\Lambda_{\mu\beta} = g_{\alpha\beta}$ , so  $(\det \Lambda)^{2} = 1$  and  $\det \Lambda = \pm 1$ . They correspond to the proper and improper Lorentz and  $\det \Lambda = \pm 1$ .

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Similarly tensors are defined as the quantities  $T_{\mu\nu\dots}{}^{\alpha\beta\gamma\dots}$  transforming under the multiple Lorentz transformation

$$\mathcal{T}_{\mu\nu\cdots}{}^{\alpha\beta\gamma\cdots} = \Lambda^{\mu'}_{\mu}\Lambda^{\nu'}_{\nu}\cdots\Lambda^{\alpha}_{\alpha'}\Lambda^{\beta}_{\beta'}\Lambda^{\gamma}_{\gamma'}\cdots(\mathcal{T}')_{\mu'\nu'\cdots}{}^{\alpha'\beta'\gamma'\cdots}$$
(72)

The Electromagnetic tensor and its dual ( $\mathbf{E} = -\nabla A^0 - \partial \mathbf{A} / \partial t$  and  $\mathbf{B} = \nabla \wedge \mathbf{A}$ )

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} = \begin{bmatrix} 0 & -E_{x} & -E_{y} & -E_{z} \\ E_{x} & 0 & -B_{z} & B_{y} \\ E_{y} & B_{z} & 0 & -B_{x} \\ E_{z} & -B_{y} & B_{x} & 0 \end{bmatrix}, F^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F^{\mu\nu\alpha\beta$$

There are two invariants:

$$F^{\mu
u}F_{\mu
u} = -{}^{*}F^{\mu
u}F_{\mu
u} = -2(\vec{E}^{2}-\vec{B}^{2}), \ F^{\mu
u}F_{\nu
ho} = g^{\mu}_{
ho}\vec{E}\cdot\vec{B}$$
(74)

The Maxwell equations (in homogenous and homogenous) and the Lorentz force are

$$\partial_{\mu}F^{\mu\nu} = \mu_{0}J^{\nu}, \ \partial_{\mu}^{*}F^{\mu\nu} = \partial_{\alpha}F^{\mu\nu} + \partial_{\mu}F^{\nu\alpha} + \partial_{\nu}F^{\alpha\mu} = 0, \ \frac{\mathrm{d}p^{\mu}}{\mathrm{d}\tau} = eF^{\mu\nu}u_{\nu}(75)$$

The Pauli matrices have the following properties:

$$\sigma^{i}\sigma^{j} = \delta^{ij} + i\epsilon^{ijk}\sigma^{k}, \ \sigma \cdot a \ \sigma \cdot b = a \cdot b + i\sigma \cdot (a \wedge b)$$
  
$$\sigma_{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (76)$$

 $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \{\sigma_i, \sigma_j\} = 2\delta_{ij}, \vec{\sigma} \times \vec{\sigma} = 2i\vec{\sigma}, \text{ they are traceless and}$ tr $\sigma_i\sigma_j = 2\delta_{ij}.$ 

#### The totally antisymmetric tensor satisfy

$$\begin{split} \epsilon^{\mu\nu\alpha\beta}\epsilon^{\mu'\nu'\alpha'\beta'} &= -g^{\mu\mu'}g^{\nu\nu'}g^{\alpha\alpha'}g^{\beta\beta'} + g^{\mu\mu'}g^{\nu\nu'}g^{\alpha\beta'}g^{\beta\alpha'} - g^{\mu\mu'}g^{\nu\beta'}g^{\alpha\nu'}g^{\alpha\nu'}g^{\alpha\mu'}g^{\alpha\mu'}g^{\mu}g^{\mu}g^{\mu'}g^{\mu}g^{\mu'}$$

where in the first case  $\sigma = \mu, \nu, \alpha$  and  $\beta$  (to form a 4 × 4 determinant), in the second case  $\sigma = \mu, \nu$  and  $\alpha$  (to form a 3 × 3 determinant), and similarly for  $\sigma'$ . The 3D version is

$$\epsilon^{ijk}\epsilon^{lmk} = \delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}, \quad \epsilon^{ikl}\epsilon^{ikm} = 2\delta^{lm}, \quad \epsilon^{ikl}\epsilon^{ikl} = 3! \quad (78)$$